Dedekind-Finite Structures

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- Determine the model-theoretic properties necessary to produce a non-well-orderable structure of a given type with a forcing or Fraenkel-Mostowski construction.
- First step: infinite Dedekind-finite sets.

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A set X is **Dedekind-finite** if X cannot be mapped bijectively onto a proper subset $Y \subset X$. Equivalently, X is Dedekind-finite if there does not exist an injection $f : \omega \longrightarrow X$.

Brief historical overview

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  The opposite implication requires the Axiom of Choice.
- Construction of infinite Dedekind-finite sets is a favorite way to show AC fails in independence proofs. Examples include: an infinite Dedekind-finite subset of $\mathbb{R}$ in Cohen's original $(\neg AC)$ model.

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Fact

$$\textit{weakly Dedekind-finite} \implies \Delta_{*\omega_1}\textit{-finite}$$

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- ω_1 is regular.

A version of Karp's Theorem

Theorem (Karp)

Let $\mathfrak{A}$ be a structure whose domain A is $\Delta_{*\omega_1}$-finite, and let $\mathscr{B}$ be a countable structure. There exists a back-and-forth family of finite partial isomorphisms between $\mathfrak{A}$ and $\mathscr{B}$ if and only if $\mathfrak{A} \equiv_{L_{\omega_1\omega}} \mathscr{B}$.

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Theorem

Let $\mathfrak{A}$ be a structure whose domain $\Delta_{*\omega_1}$-finite, and let $\mathscr{A}$ be a countable structure such that $\mathfrak{A} \equiv_{L_{\omega_1\omega}} \mathscr{A}$. Then, there exists a generic extension of the set theoretical universe $\mathscr{M}$ in which $\mathfrak{A}$ is countable and isomorphic to $\mathscr{A}$.

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  - if $n < \omega$ and $p(x_1, \ldots, x_n)$ is a non-principal n-type of $T_\delta(\mathfrak{A})$ realised in $\mathfrak{A}$, then $\bigwedge\{F(\bar{x}) : F(\bar{x}) \in p\}$ belongs to the fragment.

Scott Sentences and Rank

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The ordinal δ above is then the **Scott rank** of $\mathfrak{A}$.
The canonical **Scott sentence** $F_{\mathfrak{A}}$ for $\mathfrak{A}$ is a sentence of $L_{\omega_1\omega}$ that asserts "I am an atomic model of $T_\delta(\mathfrak{A})$", where δ is the Scott rank of $\mathfrak{A}$.

Existence of Countable Companions

Theorem (Companion Existence)

Let $\mathfrak{A}$ be a $\Delta_{*\omega_1}$-structure, and let $F_{\mathfrak{A}}$ be its Scott sentence. Then there exists a countably infinite model $\mathscr{A}$, unique up to isomorphism, that shares the Scott sentence $F = F_{\mathscr{A}} = F_{\mathfrak{A}}$, has the same Scott rank, and $\mathfrak{A} \equiv_{L_{\omega_1\omega}} \mathscr{A}$.

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We call $\mathscr{A}$ the **countable companion model** of $\mathfrak{A}$.
Let $\mathrm{Aut}(\mathscr{A})$ be the automorphism group of $\mathscr{A}$ in a "nice" set theoretical universe that contains $\mathscr{A}$ (say, $L[\mathscr{A}]$). We refer to $\mathrm{Aut}(\mathscr{A})$ as the **companion automorphism group** $\mathrm{ComAut}(\mathfrak{A})$ of $\mathfrak{A}$.

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Corollary

Let $\mathfrak{A}$ be $\Delta_{*\omega_1}$-finite. Then $|\mathrm{ComAut}(\mathfrak{A})| = 2^{\aleph_0}$.

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Fact (Rephrasing of above fact)

A $\Delta_{*_{\omega_1}}$ structure either has a weakly Dedekind-finite subset, or is Mostowski finite (i.e. any linearly ordered subset is finite).

Forcing

We construct a model of ZF that contains a model $\mathfrak{A}$ having Scott sentence $F_{\mathfrak{A}}$ in which the only relations are those that are parametrically definable. We identify the properties necessary so that $\mathfrak{A}$ will be Dedekind-finite. The precise cardinality depends on the model theoretic properties of T.

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Theorem

Let N the model constructed above.

1. *If the $L_{\omega_1\omega}$-definable closure of $\mathscr{A}$ is locally finite, and $\mathscr{A}$ is $\aleph_0$-homogeneous, then $\mathfrak{A}$ is $\Delta_{*\omega_1}$-finite.*
2. *If $\mathscr{A}$ is (first-order) $\aleph_0$-categorical, then $\mathfrak{A}$ is weakly Dedekind-finite.*