Constructibility of Potentially Isomorphic Pairs in Homogeneous Model Theory.

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Abstract

Our aim is to generalize to a non-elementary model-theoretic setting a result of Friedman, Hyttinen, and Rautila [4] which tied first-order model theoretic classification theory to constructibility of potentially isomorphic pairs under the assumption of $0^\#$. The original result stated:

Theorem. *Assume $0\#$ exists and let T be a constructible first-order theory which is countable in the constructible universe $\mathbb{L}$. Let κ be a cardinal in $\mathbb{L}$ larger than $(\aleph_1)^{\mathbb{L}}$. Then the collection of constructible pairs of models A, B of T, with $|A|, |B| = \kappa$, which are isomorphic in a cardinal- and real-preserving extension of L is itself constructible if and only if T is classifiable (i.e. superstable with NDOP and NOTOP).*

We present a generalization to Homogeneous Model Theory, a setting chosen because of its already well developed structure/non-structure theory.

Homogeneous Model Theory

Homogeneous model theory, introduced as "finite diagrams stable in power" [3], is motivated by the desire to classify the class of models of an $\mathcal{L}_{\gamma^+\omega}$ sentence ψ, with $\preccurlyeq_{\mathcal{L}_{\gamma^+\omega}}$ as the substructure relation. So the class of models will be "well behaved", we assume that the class satisfies the amalgamation property. It was proved in [3] that this is equivalent to considering the class of elementary submodels of a homogeneous monster model $\mathfrak{M}$.

$0^\#$ and Extensions of $\mathbb{L}$

If there exists a non-trivial elementary embedding of $\mathbb{L}$ into itself, then there is a closed unbounded proper class of ordinals that are indiscernible for the structure $(\mathbb{L}, \in)$. Then $0^\#$ is defined to be the real number that codes in the canonical way the Gödel numbers of the true formulas about the indiscernibles in $\mathbb{L}$.

The existence of such an elementary embedding is independent from the axioms of set theory. Moreover, the object $0^\#$ is highly nonconstructible. Throughout the work represented on this poster, we assume that $0^\#$ exists. By a **cardinal- (and real-) preserving extension** of Gödel's constructible universe $\mathbb{L}$ we mean a transitive model satisfying the Axiom of Choice, containing all the ordinals, contained in a set-generic extension of V, and which has the same cardinals (and real numbers) as $\mathbb{L}$.

References

[1] J. Baldwin, M. Laskowski, and S. Shelah Forcing Isomorphism In *Journal of Symbolic Logic 58(4) 1993, 1291-1301*

[2] S. Friedman Cardinal-Preserving Extensions In *Journal of Symbolic Logic 68(4) 2003, 1163-1170*

[3] S. Shelah Finite diagrams stable in power In *Annals of Mathematical Logic 2 (1970), 69-118*

[4] S. Friedman, T. Hyttinen, and M. Rautila Classification theory and $0^\#$ In *Journal of Symbolic Logic 68(2) 2003, 580-588*

[5] T. Hyttinen and S. Shelah Main gap for locally saturated elementary submodels of a homogeneous structure In *Journal of Symbolic Logic 66(3) 2001, 1286-1302*

[6] T. Hyttinen and H. Tuuri Constructing strongly equivalent nonisomorphic models for unstable theories In *Annals of Pure and Applied Logic 52 (1991) 203-248*

Main Result

Assume $0^\#$ exists. Suppose $\mathcal{L} \in \mathbb{L}$ is a signature such that $(|\mathcal{L}| \leqslant \omega)^{\mathbb{L}}$.

- Let $D \in \mathbb{L}$ be a **stable** finite diagram of signature $\mathcal{L}$. Let $\mu > |D|$ be a sufficiently large cardinal, and let $\mathfrak{M}$ be a homogeneous monster model of cardinality μ for the finite diagram D. Let π be a regular cardinal $> \lambda_r(\mathfrak{M})$.

  Then, the collection of pairs of constructible elementary submodels of $\mathfrak{M}$ of domain π which are isomorphic in a cardinal- and real-preserving extension of $\mathbb{L}$ is constructible if and only if $\mathfrak{M}$ is superstable and does not have $\lambda(\mathfrak{M})$-dop.
- Let $D \in \mathbf{L}$ be a good, **unstable** finite diagram of signature $\mathcal{L}$. Let $\mu = \mu^{<\mu} > |D|$ be a sufficently large cardinal, and let $\mathfrak{M}$ be a homogeneous monster model of cardinality μ for the finite diagram D. Let $\xi > |\mathcal{L}|$ be a regular uncountable cardinal, and let π be such that $\pi > 2^{<\xi}$.

  Then, the collection of pairs of constructible elementary submodels of $\mathfrak{M}$ of domain π which are isomorphic in a cardinal- and real-preserving extension of $\mathbb{L}$ is equiconstructible with $0^\#$.

Similarly to the first-order case, here the dividing lines coincide precisely with those in the proof of the Main Gap for locally saturated models in Homogeneous Model Theory [5].

Model Theoretic Methods

Trees or stationary sets are "coded in" to models in question, with the method depending on the stability-theoretic case in hand. Below are brief sketches of the main ideas.

In the **unstable case**, we make use of the existence of a model with no *universal equivalence tree*. For t a tree, the *Ehrenfeucht-Fraïssé game approximated by* t between models $\mathscr{A}$ and $\mathscr{B}$, $G^t(\mathscr{A}, \mathscr{B})$, is the following: At each move α, (i) $\forall$ chooses $x_\alpha \in t$, and either $a_\alpha \in \mathscr{A}$ or $b_\alpha \in \mathscr{B}$. (ii) if $\forall$ chooses a point in $\mathscr{A}$, $\exists$ chooses $b_\alpha \in \mathscr{B}$ and vice versa.

Player $\forall$ must move so that $x_\beta)_{\beta \leqslant \alpha}$ forms a strictly increasing sequence in t. Player $\exists$ must move so that $\{(a_\beta, b_\beta) : \beta \leqslant \alpha\}$ is a partial isomorphism $\mathscr{A} \longrightarrow \mathscr{B}$. He who breaks the rules first loses. If $\exists$ has a winning strategy we say $\mathscr{A} \cong^t \mathscr{B}$. If $|\mathscr{A}| = \kappa$ and t is a tree such that for all $|\mathscr{B}|$ of size κ, $\mathscr{A} \cong^t \mathscr{B} \Rightarrow \mathscr{A} \cong \mathscr{B}$, then t is a universal equivalence tree of $\mathscr{A}$.

Using similar reasoning to that in [6], for regular uncountable cardinals $\kappa = \kappa^{<\kappa}$, there is a model $\mathscr{A} \preccurlyeq \mathfrak{M}$ of size κ which does not have a universal equivalence tree on κ of height κ without κ-branches. This means that for every such tree u, there is a model $\mathcal{B} \preccurlyeq \mathfrak{M}$ of size κ such that $\mathscr{A} \cong^u \mathscr{A}$ but $\mathscr{A} \not\cong \mathscr{B}$. However, (very roughly speaking) in an extension where the tree receives a κ-branch, the two models will be isomorphic.

In the **strictly stable** case, we cannot rely on the universal equivalence tree method. Instead, we construct two trees $t_1 \subset t_2$ differing in that t_2 has a portion that depends on the properties of a stationary set S. In an extension of the universe which shoots a club through the stationary set S, the two trees become identical.

We perform primary model constructions along the two trees. We show that the two resulting models are not isomorphic, but that in extensions which kill stationarity of S, the two models become isomorphic.

The case of **superstable with** $\lambda(\mathfrak{M})$**-dop**, is similar to the strictly stable case, but is very technical.

In the case of **superstable without** $\lambda(\mathfrak{M})$**-dop**, the class of models is sufficiently nicely definable, thus any two models will be isomorphic in a cardinal- and real-preserving extension of $\mathbb{L}$ iff they are isomorphic in $\mathbb{L}$ (see [1]).

Set Theoretic Tools

We have the following notion of **reduction** of sets to other sets of known constructible complexity:

Suppose $\langle X_0, X_1\rangle$, $\langle Y_0, Y_1\rangle$ are pairs of disjoint collections of constructible sets. Note that $\langle X_0, X_1\rangle$ and $\langle Y_0, Y_1\rangle$ need not be constructible themselves. We say that $\langle X_0, X_1\rangle$ reduces to $\langle Y_0, Y_1\rangle$ if there exists a constructible function $g \in \mathbb{L}$ such that $x \in X_0 \Rightarrow g(x) \in Y_0$ and $x \in X_1 \Rightarrow g(x) \in Y_1$.

We write X_0 instead of $\langle X_0, X_1\rangle$ if X_0 is the complement of X_1 within some constructible set – analogously for the Ys.

Let λ be an infinite cardinal in $\mathbb{L}$ and $(\lambda^+ = \kappa)^{\mathbb{L}}$. In [2] it was shown that $0^\#$ is equiconstructible with the set of constructible *trees on κ of height κ which have a κ-branch in some cardinal-preserving extension of* $\mathbb{L}$. This uses a method where guesses about membership in $0^\#$ are killed using class forcing, or in other terms, by killing guesses of finite sequences of Silver indiscernibles. These trees are then reduced to "nicely-definable trees" using a technical argument.

Then, these "nicely definable trees" are reduced to the collection of sets $S \subseteq (S^\kappa_\omega)^{\mathbb{L}}$ *which are stationary in* $\mathbb{L}$, *but which cease to be stationary in some cardinal-preserving extension of* $\mathbb{L}$. The argument uses a gap-1 morass.

Open Questions

- Can this be extended to other non-elementary contexts? (For excellent classes, most likely. Others?)
- Can the limitation to locally saturated models in the stable cases be done away with?
- Can these methods point the way to other tools for classifying models in contexts which are sensitive to the set-theoretic universe in which considerations take place?