Constructibility of Potentially Isomorphic Pairs in Homogeneous Model Theory.

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Main Result

Main Theorem. *Denote by $\mathbb{L}$ Gödel's constructible universe. Let $\mathfrak{M} \in \mathbb{L}$ be a large stable homogeneous monster model in a countable in $\mathbb{L}$ similarity type $\mathcal{L}$.*
Assume $(\pi = \mathrm{cf}(\pi) > \lambda_r(\mathfrak{M}))^{\mathbb{L}}$. Let $PIP_\pi^{\lambda_r(\mathfrak{M})}$ be the collection of pairs $(\mathscr{A}, \mathscr{B}) \in \mathbb{L}$ of locally $\mathbf{F}^{\mathfrak{M}}_{\lambda_r(\mathfrak{M})}$-saturated elementary substructures of $\mathfrak{M}$ with universe π such that there is a cardinal- and $\mathscr{P}(\lambda_r(\mathfrak{M}))$-preserving extension of $\mathbb{L}$ in which $\mathscr{A} \cong \mathscr{B}$.
Then $PIP_\pi^{\lambda_r(\mathfrak{M})}$ is constructible $\Leftrightarrow$ $\mathfrak{M}$ is superstable and does not have $\lambda(\mathfrak{M})$-dop.

This is a generalization of [5] to a non-elementary model-theoretic setting. Similarly, cases coincide with those in the respective Main Gap Theorem [6]. We "code in" trees or stationary sets to models in question, with the method depending on the stability-theoretic case. Proofs in our strictly stable and dop cases are much more complicated, owing to unavailability of Ehrenfeucht-Mostowski (EM) constructions.

Homogeneous Model Theory

Homogeneous model theory, introduced in [4], is motivated by the desire to classify the class of models of an $\mathcal{L}_{\gamma^+\omega}$ sentence ψ, with $\preccurlyeq_{\mathcal{L}_{\gamma^+\omega}}$ as the substructure relation. So the class of models will be "well behaved", we assume that the class satisfies the amalgamation property. It was proved in [4] that this is equivalent to considering the class of elementary submodels of a homogeneous monster model $\mathfrak{M}$.

$0^\#$ and Extensions of $\mathbb{L}$

If there is a non-trivial elementary embedding of Gödel's constructible universe $\mathbb{L}$ into itself, then there is a closed unbounded proper class of ordinals that are indiscernible for the structure $(\mathbb{L}, \in)$. Then $\mathbf{0}^\#$ is defined to be the real number that codes in the canonical way the Gödel numbers of the true formulas about the indiscernibles in $\mathbb{L}$.
The existence of $0^\#$ is independent from ZFC. Moreover, $0^\#$ is highly non-constructible. We assume throughout that $0^\#$ exists.
By a **cardinal- (and $\mathscr{P}(\kappa)$-) preserving extension** of $\mathbb{L}$ we mean a transitive model satisfying AC, containing all the ordinals, contained in a set-generic extension of V, and which has the same cardinals (and subsets of κ) as $\mathbb{L}$.

References

[1] J. Baldwin, M. Laskowski, and S. Shelah Forcing Isomorphism. In *Journal of Symbolic Logic 58(4) 1993, 1291-1301.*

[2] S. Friedman Cardinal-Preserving Extensions. In *Journal of Symbolic Logic 68(4) 2003, 1163-1170.*

[3] S. Friedman, T. Hyttinen, A. Walczak-Typke Potential Isomorphism of Elementary Substructures of a Strictly Stable Homogeneous Model. Submitted *22 March 2010.*

[4] S. Shelah Finite diagrams stable in power. In *Annals of Mathematical Logic 2 (1970), 69-118.*

[5] S. Friedman, T. Hyttinen, and M. Rautila Classification theory and $0^\#$. In *Journal of Symbolic Logic 68(2) 2003, 580-588.*

[6] T. Hyttinen and S. Shelah Main gap for locally saturated elementary submodels of a homogeneous structure. In *Journal of Symbolic Logic 66(3) 2001, 1286-1302.*

[7] T. Hyttinen and H. Tuuri Constructing strongly equivalent nonisomorphic models for unstable theories. In *Annals of Pure and Applied Logic 52 (1991) 203-248.*

Unstable Case

The **unstable case** is similar to the first-order case, relying on the order property, and using an EM model construction.

Theorem. *Let $\mathfrak{M} \in \mathbb{L}$ be a sufficiently large unstable homoegeneous monster. Let λ be an infinite $\mathbb{L}$-cardinal, $(\lambda^+ = \pi)^{\mathbb{L}}$, and let Ψ be the EM template. Let $\mathscr{A} = EM^{sk}(\varsigma_\pi, \Psi)\restriction_{\mathcal{L}}$. Let $PI\mathscr{A}^\lambda_\pi$ be the collection of elementary submodels $\mathscr{B}$ of $\mathfrak{M}$ with universe π such that there is a cardinal- and $\mathscr{P}(\lambda)$-preserving extension of $\mathbb{L}$ in which $\mathscr{A} \cong \mathscr{B}$.*
Then

$$0^\# \xrightarrow{\mathbb{L}} PI\mathscr{A}^\lambda_\pi.$$

We make use of the existence of a model with no **universal equivalence tree**. For t a tree, the **Ehrenfeucht-Fraïssé game approximated by** t between models $\mathscr{A}$ and $\mathscr{B}$, $G^t(\mathscr{A}, \mathscr{B})$, is the following: At each move α, (i) $\forall$ chooses $x_\alpha \in t$, and either $a_\alpha \in \mathscr{A}$ or $b_\alpha \in \mathscr{B}$. (ii) if $\forall$ chooses a point in $\mathscr{A}$, $\exists$ chooses $b_\alpha \in \mathscr{B}$ and vice versa. Player $\forall$ must move so that $(x_\beta)_{\beta \leqslant \alpha}$ forms a strictly increasing sequence in t. Player $\exists$ must move so that $\{(a_\beta, b_\beta) : \beta \leqslant \alpha\}$ is a partial isomorphism $\mathscr{A} \longrightarrow \mathscr{B}$. He who breaks the rules first loses. If $\exists$ has a winning strategy we say $\mathscr{A} \cong^t \mathscr{B}$. If $|\mathscr{A}| = \kappa$ and t is a tree such that for all $|\mathscr{B}|$ of size κ, $\mathscr{A} \cong^t \mathscr{B} \Rightarrow \mathscr{A} \cong \mathscr{B}$, then t is a universal equivalence tree of $\mathscr{A}$.
As in [7], for regular uncountable cardinals $\kappa = \kappa^{<\kappa}$ there is a model $\mathscr{A} \preccurlyeq \mathfrak{M}$ of size κ which does not have a universal equivalence tree on κ of height κ without κ-branches. Thus for every such tree u, there is a model $\mathcal{B} \preccurlyeq \mathfrak{M}$ of size κ such that $\mathscr{A} \cong^u \mathscr{A}$ but $\mathscr{A} \not\cong \mathscr{B}$. However, in an extension where the tree receives a κ-branch, the two models will be isomorphic.

Strictly Stable Case

The proof of the **strictly stable** case [3] is essentially different from the first-order context. We cannot find tree indiscernibles without assuming large cardinals, thus we cannot use EM constructions. Our method uses a non-linearly ordered skeleton, thus is applicable in other contexts.

Theorem. *Let $\mathfrak{M} \in \mathbb{L}$ be a sufficiently large strictly stable homogeneous monster.*
Let π be such that $\pi = \mathrm{cf}(\pi) > \lambda_r(\mathfrak{M})$. Let $PIP_\pi^{\lambda_r(\mathfrak{M})}$ be the collection of pairs $(\mathscr{A}, \mathscr{B}) \in \mathbb{L}$ of locally $\mathbf{F}^{\mathfrak{M}}_{\lambda_r(\mathfrak{M})}$-saturated elementary substructures of $\mathfrak{M}$ with universe π such that there is a cardinal- and $\mathscr{P}(\lambda_r(\mathfrak{M}))$-preserving extension of $\mathbb{L}$ in which $\mathscr{A} \cong \mathscr{B}$.
Then, $PIP_\pi^{\lambda_r(\mathfrak{M})}$ is equiconstructible with $0^\#$.

We construct two trees $t_1 \subset t_2$ differing in that t_2 has a portion that depends on the properties of a stationary set S. In an extension of the universe which shoots a club through the stationary set S, the two trees become identical.
We perform primary model constructions along the two trees. We show that the two resulting models are not isomorphic, but that in extensions which kill stationarity of S, the two models become isomorphic. If S remains stationary in the extension, then the two models remain nonisomorphic. Equiconstructibility with $0^\#$ follows from "Set Theoretic Tools" below, and other results from [2].

Superstable with $\lambda(\mathfrak{M})$-dop

The case of **superstable with $\lambda(\mathfrak{M})$-dop**, is extremely technical. As in non-structure theorems for first order stable theories with DOP, we must use primary model constructions. To prove this case, we needed to advance knowledge, among other things, of the existence of certain free extensions in the homogeneous setting, as well as extend results about reducibility of stationary sets of higher cofinality.

Structure: Superstable without $\lambda(\mathfrak{M})$-dop

Theorem. *(i) Suppose $\mathfrak{M}$ is a superstable homogeneous monster model without $\lambda(\mathfrak{M})$-dop. Suppose that $(\pi > \lambda(\mathfrak{M}))^{\mathbb{L}}$ is regular in $\mathbb{L}$. If $\mathscr{A}$ and $\mathscr{B}$ are locally $\lambda(\mathfrak{M})$-saturated elementary submodels of $\mathfrak{M}$ and $\mathscr{A} \equiv_{\infty\pi} \mathscr{B}$, then $\mathscr{A} \cong \mathscr{B}$.*
(ii) Let $PIP_\pi^{\lambda_r(\mathfrak{M})}$ be the collection of pairs $(\mathscr{A}, \mathscr{B}) \in \mathbb{L}$ of of locally $\lambda(\mathfrak{M})$-saturated elementary submodels of $\mathfrak{M}$ with universe π such that there is a cardinal and $\mathscr{P}(\lambda_r(\mathfrak{M}))$-preserving extension of $\mathbb{L}$ in which $\mathscr{A} \cong \mathscr{B}$. Then $PIP_\pi^{\lambda_r(\mathfrak{M})}$ is constructible.

The proof of (i) goes just like the proof of Shelah, *Classification Theory*, Ch XIII, Thm 1.1. Note that by [6], the decompositions in the proof can be constructed so that they are not only regular s-free trees but also so that the following holds: If $t, u, w \in P$ and $u^- = w^- = t$ then either $\mathrm{tp}(f(u)/g(t)) = \mathrm{tp}(f(w)/g(t))$ or $\mathrm{tp}(f(u)/g(t))$ is orthogonal to $\mathrm{tp}(f(w)/g(t))$.
To see (ii), note that, just as in [5], any two elementary submodels of $\mathfrak{M}$ are isomorphic in a cardinal- and $\mathscr{P}(\lambda_r(\mathfrak{M}))$-preserving extension of $\mathbb{L}$ if and only if they are isomorphic in $\mathbb{L}$ itself, using arguments from [1].

Set Theoretic Tools

Definition. Suppose $\langle X_0, X_1\rangle$, $\langle Y_0, Y_1\rangle$ are (not necessarily constructible) pairs of disjoint collections of constructible sets. We write that $\langle X_0, X_1\rangle \xrightarrow{\mathbb{L}} \langle Y_0, Y_1\rangle$ if there exists a function $g \in \mathbb{L}$ such that $x \in X_0 \Rightarrow g(x) \in Y_0$ and $x \in X_1 \Rightarrow g(x) \in Y_1$.
We write X_0 instead of $\langle X_0, X_1\rangle$ if X_0 is the complement of X_1 within some constructible set – analogously for the Ys.

Theorem ([2])**.** *Let λ be an infinite cardinal in $\mathbb{L}$ and $(\lambda^+ = \kappa)^{\mathbb{L}}$. Then $0^\#$ is equiconstructible with the set of constructible* ***trees on κ of height κ which have a κ-branch in some cardinal-preserving extension of*** *$\mathbb{L}$.*
Denote by $\mathcal{S}(\kappa)$ is the collection of sets $S \subseteq (S^\kappa_\omega)^{\mathbb{L}}$ ***which are stationary in $\mathbb{L}$, but which cease to be stationary in some cardinal-preserving extension of*** *$\mathbb{L}$. Then, $0^\# \xrightarrow{\mathbb{L}} \mathcal{S}(\kappa)$.*