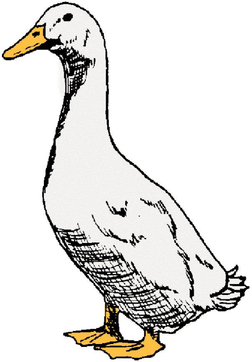

FORCING & CH
IN A NUTSHELL

Set Theory and Models
  Axioms and Notation
  Models of Set Theory

Cohen's Forcing
  The Intuition Behind Forcing
  Forcing

An Example of a Forcing Model

Set Theory and Models



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Axioms of ZFC (cont.)

6. **Axiom (Schema) of separation (or subset axiom):** Given any set and any formula $P(x)$ in the language of set theory with at least one free variable, there is a subset of the original set containing precisely those elements x for which $P(x)$ holds.

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10. **Axiom of choice:** Every family of non-empty sets has a choice function.

Some Notation

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- We denote the **ordered pair** $\{\{x\}, \{x, y\}\}$ as $\langle x, y \rangle$.

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- We use small Greek letters for the **ordinal numbers**. The proper class containing all ordinals is denoted *On*.
- We identify **cardinal numbers** with the smallest ordinal having that cardinality. Thus $|\omega| = \aleph_0$ and $|\omega_\alpha| = \aleph_\alpha$.
- $|\mathbb{R}| = |P(\omega)| = 2^{\aleph_0}$. The **Continuum Hypothesis (CH)** is the conjecture that
$$2^{\aleph_0} = \aleph_1.$$

Definition

We take the **language of set theory** to be $\mathscr{L} = \{\in\}$. This language is first-order.
A **model of set theory** is a class $\mathscr{M}$ and a binary relation E on $\mathscr{M}$, together with an interpretation of the language of set theory: We define

$$\mathscr{M} \models \phi$$

(read: "$\mathscr{M}$ **satisfies** ϕ") inductively on the complexity of ϕ:

$$\mathscr{M} \models x \in y \text{ iff } xEy,$$
$$\mathscr{M} \models \forall x \in \phi \text{ iff for all } x \in \mathscr{M}, \mathscr{M} \models \phi(x),$$
$$etc.$$

$\mathscr{M}$ is a **model of ZF** if it satisfies all the axioms of ZF.

We will take E to be the standard $\in$.

Gödel's Results

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Since we assume ZF is consistent, we *assume* ZF has a model.

Properties of a Model of Set Theory

Downward Löwenheim-Skolem Theorem

If a set of sentences Σ *in a first order language has a model, then it has a model* $\mathfrak{A}$ *such that* $|\mathfrak{A}| \leqslant \max(|\Sigma|, \aleph_0)$*.*

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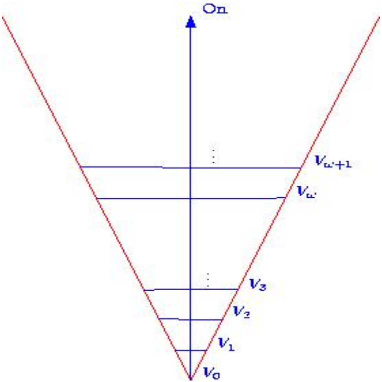

By the Axiom of Regularity, every set in a model of set theory has a rank:

Definition

Let $V_0 = \emptyset$, $V_{\alpha+1} = P(V_\alpha)$, and for γ a limit ordinal, $V_\gamma = \bigcup_{\alpha<\gamma} V_\alpha$. The **universe (or domain)** of a model of set theory $V = \bigcup_{\alpha\in On} V_\alpha$. The **rank** of a set X, $rk(X)$, is the smallest α such that $X \in V_{\alpha+1}$.

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x \setminus y, & x \times y, & domain(x), \\
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$y = P(x)$ and “x is a cardinal” are NOT ABSOLUTE!

The Constructible Universe

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L is the smallest possible transitive model of set theory. Hence, if we want other examples of models, we'll have to build extensions.

Forcing

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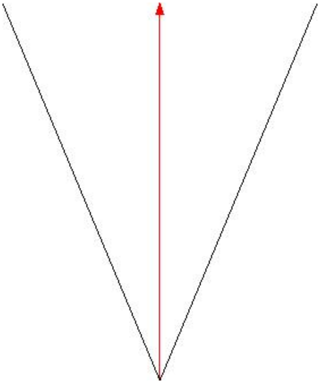

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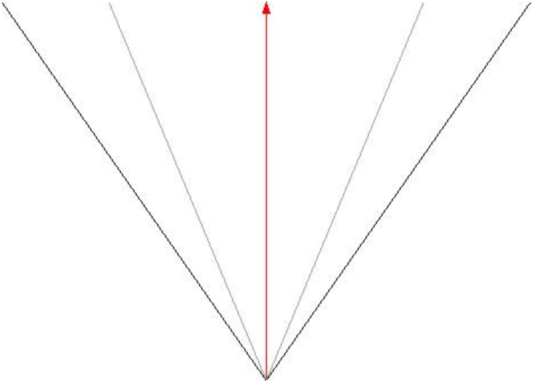

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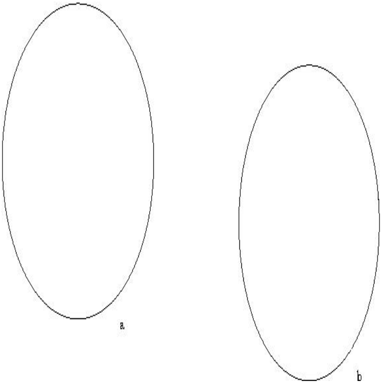

M is a model of ZFC. Let $a, b \in M$, where a is infinite, and $b \neq 0$.

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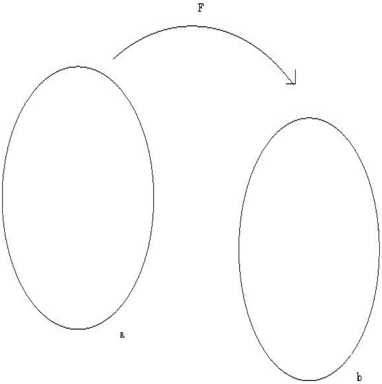

We want a mapping $F : a \longrightarrow b$, but $F \notin M$.
We will extend the model M to a model N such that $F \in N$.

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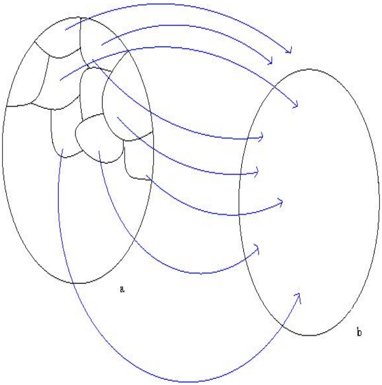

Let C be all the mappings from finite subsets of a to b. Note that $C \in M$

Let G be the set of all finite restrictions of F. Then $\bigcup G = F$.

Outside of M, $G \subset C$, but $M \not\models G \subset C$. So, $G \notin M$.

We must somehow select this G out of all the possible elements of C, and then use G to build a fat model that contains G and $F = \bigcup G$.

Two Trivial Remarks about G and C

Suppose we are given C and F.

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- If q is an extension of the function p, the condition q gives more information that p. We then write $q \leq p$. This makes C a partially ordered set, with the trivial function 0 as the $\leq$-largest element.

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With these properties, $F = \bigcup G$ will be a mapping from a **subset** of a onto a **subset** of b.

But we want $dom(F) = a$ and $rng(F) = b$!

G must intersect all sets of the form

$$\{p : x \in dom(p)\}$$

for $x \in a$, and

$$\{p : y \in rng(p)\}$$

for $y \in b$. These sets are in M.
These sets have the property that any p has an extension in each of these sets. We require G to intersect every set in M with this property.

Notions of Forcing, and Density

Definitions

A **notion of forcing** is a partially ordered set C having a largest element. We write $\leq_C$ for the ordering, and 1_C for the largest element. The elements of C are called **conditions**. If $p \leq_C q$, we way p is an **extension** of q. A subset D of C is **dense** in C if every condition in C has an extension in D.

Generic Sets

Definition

Let C be a notion of forcing, and let M be a set. A subset G of C is **generic** in C over M if the following conditions hold.

1. $1_C \in G$.
2. For all $p \in G$ and $q \geq_c p$, $q \in G$.
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Theorem (Existence Theorem)

Let C be a notion of forcing, M a countable set. Let $p \in C$. Then there is a set G which is generic in C over M and contains p.

Definition

Let $\in_G$ be a relation on the set M defined by

$$a \in_G b \leftrightarrow (\exists p \in G)(\langle a, p \rangle \in b).$$

Note:

$$a \in_G b \rightarrow rk(a) < rk(b).$$

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Theorem (Main Theorem)

Let M be a countable model of ZFC, C a notion of forcing in M, G a set which is generic in C over M. Then $M[G]$ is a countable

An Example of a Forcing Model of $ZF + \neg CH$

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Let G be generic in C over M, and let $M[G]$ be the resulting forcing model.

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When does $\aleph_{42}^{M} = \aleph_{42}^{M[G]}$?

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Theorem

If C satisfies the countable chain condition in M, then M and $M[G]$ have the same cardinals, and if $\alpha \in M$, then $cf^{M}(\alpha) = cf^{M[G]}(\alpha)$.