Algebraic Structures in Set Theory without the Axiom of Choice

Agatha Walczak-Typke

Leeds University

July 28, 2004

Model Theoretic Notions

Definition

A theory T is said to be $\aleph_0$-**categorical** if it has, up to isomorphism, exactly one model of cardinality $\aleph_0$.

Model Theoretic Notions

Definition

A theory T is said to be $\aleph_0$-**categorical** if it has, up to isomorphism, exactly one model of cardinality $\aleph_0$.

Definition

A complete theory T is said to be **strongly minimal** if every parametrically definable subset of each of its models is either finite or co-finite.

Model Theoretic Notions

Definition

A theory T is said to be $\aleph_0$-**categorical** if it has, up to isomorphism, exactly one model of cardinality $\aleph_0$.

Definition

A complete theory T is said to be **strongly minimal** if every parametrically definable subset of each of its models is either finite or co-finite.

Examples

1. $(\mathbb{C}, =, +, \times)$ is strongly minimal
2. An $\aleph_0$-dimensional vector space over $\mathbb{F}_p$ is $\aleph_0$-categorical strongly minimal.

Let $S_1(A)$ denote the set of all complete types of formulae in one variable with parameters from A.

Definition

A model M is $\aleph_0$-**saturated** if for all finite $A \subset M$, M realizes every type in $S_1(A)$.

Let $S_1(A)$ denote the set of all complete types of formulae in one variable with parameters from A.

Definition

A model M is $\aleph_0$-**saturated** if for all finite $A \subset M$, M realizes every type in $S_1(A)$.

Definition

Let T be a complete theory (of any cardinality) with an infinite model. T is said to be $\aleph_0$-**stable** if for all $\mathcal{M} \models T$ and $A \subset M$ of cardinality at most $\aleph_0$, $|S_1(A)| \leqslant \aleph_0$

Model Theoretic Notions (cont.)

Definition

Let T be a complete theory in a language L, and let $\mathfrak{C}$ denote a saturated model of T of sufficiently large cardinality. The relation $\mathrm{MR}(\varphi) = \alpha$, for φ a formula in n variables and α an ordinal, -1, or ∞ is defined by the following recursion.

(i) $\mathrm{MR}(\varphi) = -1$ if φ is inconsistent;

(ii) $\mathrm{MR}(\varphi) = \alpha$ if $\mathrm{MR}(\varphi) \not< \alpha$ and there is a finite $n \in \omega$ such that there are L-formulae $\varphi_i(\bar{x})$, $0 \leqslant i \leqslant n$ with parameters from $\mathfrak{C}$, such that the sets $\varphi(\mathfrak{C}^n) \cap \varphi_i(\mathfrak{C}^n)$, are pairwise disjoint and $\mathrm{MR}(\varphi \wedge \varphi_i) < \alpha$ for at least one $i < n$;

This rank is called **Morley rank**.

Model Theoretic Notions (cont.)

Definition

Let T be a complete theory in a language L, and let $\mathfrak{C}$ denote a saturated model of T of sufficiently large cardinality. The relation $\mathrm{MR}(\varphi) = \alpha$, for φ a formula in n variables and α an ordinal, -1, or ∞ is defined by the following recursion.

(i) $\mathrm{MR}(\varphi) = -1$ if φ is inconsistent;

(ii) $\mathrm{MR}(\varphi) = \alpha$ if $\mathrm{MR}(\varphi) \not< \alpha$ and there is a finite $n \in \omega$ such that there are L-formulae $\varphi_i(\bar{x})$, $0 \leqslant i \leqslant n$ with parameters from $\mathfrak{C}$, such that the sets $\varphi(\mathfrak{C}^n) \cap \varphi_i(\mathfrak{C}^n)$, are pairwise disjoint and $\mathrm{MR}(\varphi \wedge \varphi_i) < \alpha$ for at least one $i < n$;

This rank is called **Morley rank**.
If T is a theory, we take $\mathrm{MR}(T)$ to be the Morley rank of the formula $x = x$ in any $\aleph_0$-saturated model of T.

Definition

A theory that contains the theory of linear orderings is said to be **o-minimal** if in all of its models the sets parametrically definable with formulas of one variable are unions of finitely many intervals.

Definition

A theory that contains the theory of linear orderings is said to be **o-minimal** if in all of its models the sets parametrically definable with formulas of one variable are unions of finitely many intervals.

Examples

1. A discrete linear order with or without endpoints in the language $L = \{=, <\}$.
2. A dense linear order with or without endpoints in the language L.
3. A divisible ordered abelian group in the language $L = \{=, +, 0, <\}$.

A Reminder about life without choice

Axiom of Choice

Every family of non-empty sets has a choice function.

A Reminder about life without choice

Axiom of Choice

Every family of non-empty sets has a choice function.

In the absence of the Axiom of Choice:

- Not all sets can be well, or even linearly, ordered.

A Reminder about life without choice

Axiom of Choice

Every family of non-empty sets has a choice function.

In the absence of the Axiom of Choice:

- Not all sets can be well, or even linearly, ordered.
- The cardinal numbers are not linearly ordered.

A Reminder about life without choice

Axiom of Choice

Every family of non-empty sets has a choice function.

In the absence of the Axiom of Choice:

- Not all sets can be well, or even linearly, ordered.
- The cardinal numbers are not linearly ordered.
- An infinite dimensional vector space need not have a basis.

Defintions of Finiteness

Definition

A set X is **finite** if there exists a bijective mapping between X and some $n \in \omega$. A set X is **infinite** if it fails the definition of finite.

Defintions of Finiteness

Definition

A set X is **finite** if there exists a bijective mapping between X and some $n \in \omega$. A set X is **infinite** if it fails the definition of finite.

Definition

A set X is **amorphous** if it cannot be expressed as the union of two disjoint infinite sets.

Defintions of Finiteness

Definition

A set X is **finite** if there exists a bijective mapping between X and some $n \in \omega$. A set X is **infinite** if it fails the definition of finite.

Definition

A set X is **amorphous** if it cannot be expressed as the union of two disjoint infinite sets.

Definition

A set X is **II-finite** if any linearly ordered partition of X is finite.

Definition

A set X is **weakly Dedekind-finite** if there does not exist a surjective mapping from X onto ω. Equivalently, X is weakly Dedekind-finite if $P(X)$ is Dedekind-finite (as defined in the abstract).

Definition

A set X is **weakly Dedekind-finite** if there does not exist a surjective mapping from X onto ω. Equivalently, X is weakly Dedekind-finite if $P(X)$ is Dedekind-finite (as defined in the abstract).

Definition

A subset of a linearly ordered set $(X, \leqslant)$ is called an *interval* if it has the form $(a, b) = \{x : a < x < b\}$, $[a, b]$, $[a, b)$, or $(a, b]$, where $a, b \in X \cup \{\pm\infty\}$. A subset is $Y \subseteq X$ is *convex* if $a < x < b \wedge a, b \in Y \Rightarrow x \in Y$. The linearly ordered set $(X, \leqslant)$ is said to be **o-amorphous** if it is infinite and its only subsets are finite unions of intervals.

Definition

A set X is **weakly Dedekind-finite** if there does not exist a surjective mapping from X onto ω. Equivalently, X is weakly Dedekind-finite if $P(X)$ is Dedekind-finite (as defined in the abstract).

Definition

A subset of a linearly ordered set $(X, \leqslant)$ is called an *interval* if it has the form $(a, b) = \{x : a < x < b\}$, $[a, b]$, $[a, b)$, or $(a, b]$, where $a, b \in X \cup \{\pm\infty\}$. A subset is $Y \subseteq X$ is *convex* if $a < x < b \wedge a, b \in Y \Rightarrow x \in Y$. The linearly ordered set $(X, \leqslant)$ is said to be **o-amorphous** if it is infinite and its only subsets are finite unions of intervals.

X is o-amorphous $\Longrightarrow X$ is weakly Dedekind-finite.

Definitions of Finiteness (cont.)

Definition

Let $\mathscr{N}$ be a model of ZF. The relation $\mathrm{MT}(X) = \alpha$, for $X \in \mathscr{N}$ and α an ordinal or -1, is defined by the following recursion.

(i) $\mathrm{MT}(X) = -1$ if $X = \emptyset$.

(ii) $\mathrm{MT}(X) = \alpha$ if $\mathrm{MT}(X) \not< \alpha$, and there is a finite n such that if $\{X_i : 0 \leqslant i \leqslant n\}$ are pairwise disjoint subsets of X, then for some i, $\mathrm{MT}(X_i) < \alpha$.

We call this rank **MT-rank**.

Definitions of Finiteness (cont.)

Definition

Let $\mathscr{N}$ be a model of ZF. The relation $\mathrm{MT}(X) = \alpha$, for $X \in \mathscr{N}$ and α an ordinal or -1, is defined by the following recursion.

(i) $\mathrm{MT}(X) = -1$ if $X = \emptyset$.

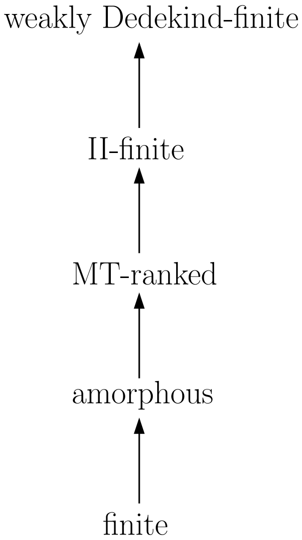
(ii) $\mathrm{MT}(X) = \alpha$ if $\mathrm{MT}(X) \not< \alpha$, and there is a finite n such that if $\{X_i : 0 \leqslant i \leqslant n\}$ are pairwise disjoint subsets of X, then for some i, $\mathrm{MT}(X_i) < \alpha$.

We call this rank **MT-rank**.

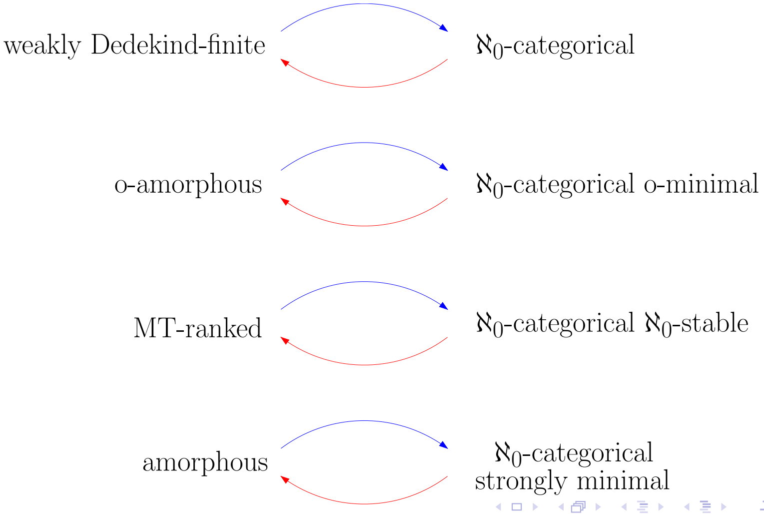
Definition

A set X of MT-rank α has **degree** k if X has k pairwise disjoint subsets of MT-rank α, but for any $k+1$ pairwise disjoint subsets, at least one of them has smaller rank.

Relations between notions of finiteness



The correlation between finiteness and model theoretic notions



Theory Analysis

Let G be an infinite set admitting an algebraic structure. (A group, field, ordering, etc.)
Let $Th(G)$ be the complete theory of G.

Theorem

1. *If G is weakly Dedekind-finite, then $Th(G)$ is $\aleph_0$-categorical.*

Theory Analysis

Let G be an infinite set admitting an algebraic structure. (A group, field, ordering, etc.)
Let $Th(G)$ be the complete theory of G.

Theorem

1. *If G is weakly Dedekind-finite, then $Th(G)$ is $\aleph_0$-categorical.*
2. *If G is o-amorphous, then $Th(G)$ is $\aleph_0$-categorical o-minimal.*

Theory Analysis

Let G be an infinite set admitting an algebraic structure. (A group, field, ordering, etc.)
Let $Th(G)$ be the complete theory of G.

Theorem

1. *If G is weakly Dedekind-finite, then $Th(G)$ is $\aleph_0$-categorical.*
2. *If G is o-amorphous, then $Th(G)$ is $\aleph_0$-categorical o-minimal.*
3. *If G has MT-rank α, then $Th(G)$ is $\aleph_0$-categorical and $\aleph_0$-stable. Thus $Th(G)$ has finite Morley rank $n \leqslant \alpha$.*

Theory Analysis

Let G be an infinite set admitting an algebraic structure. (A group, field, ordering, etc.)
Let $Th(G)$ be the complete theory of G.

Theorem

1. *If G is weakly Dedekind-finite, then $Th(G)$ is $\aleph_0$-categorical.*
2. *If G is o-amorphous, then $Th(G)$ is $\aleph_0$-categorical o-minimal.*
3. *If G has MT-rank α, then $Th(G)$ is $\aleph_0$-categorical and $\aleph_0$-stable. Thus $Th(G)$ has finite Morley rank $n \leqslant \alpha$.*
4. *If G is amorphous, then $Th(G)$ is $\aleph_0$-categorical, strongly minimal.*

Conclusions from the analysis

The possible structures are limited by the theory:

- There are no infinite amorphous fields.
- Infinite amorphous groups are elementary abelian p-groups.
- Groups with MT-rank are abelian-by-finite.
- etc.

Forcing

We construct a model of ZF that contains a model $\mathcal{A}$ of a first order relational theory T in which the only relations are those that are parametrically definable. The set $\mathcal{A}$ will be Dedekind-finite, the precise cardinality depending on the model theoretic properties of T.

Let $M \models ZF + V = L$. Assume M contains an infinite set A that admits finitely many relations and models a first order relational theory T.

Let $M \models ZF + V = L$. Assume M contains an infinite set A that admits finitely many relations and models a first order relational theory T.
We add generic sets of reals indexed by A and the set that contains them to M. Let N be the resulting model.

Let $M \models ZF + V = L$. Assume M contains an infinite set A that admits finitely many relations and models a first order relational theory T.

We add generic sets of reals indexed by A and the set that contains them to M. Let N be the resulting model. Let $\mathcal{A}$ be the set of sets of generic reals in N. $\mathcal{A}$ has the relational structure inherited from the indexing, though the indexing itself is not in N.

Let $M \models ZF + V = L$. Assume M contains an infinite set A that admits finitely many relations and models a first order relational theory T.
We add generic sets of reals indexed by A and the set that contains them to M. Let N be the resulting model. Let $\mathcal{A}$ be the set of sets of generic reals in N. $\mathcal{A}$ has the relational structure inherited from the indexing, though the indexing itself is not in N.

Theorem

Let N the model constructed above. If T is

1. *$\aleph_0$-categorical strongly minimal, then $\mathcal{A}$ is amorphous.*

Let $M \models ZF + V = L$. Assume M contains an infinite set A that admits finitely many relations and models a first order relational theory T.

We add generic sets of reals indexed by A and the set that contains them to M. Let N be the resulting model. Let $\mathcal{A}$ be the set of sets of generic reals in N. $\mathcal{A}$ has the relational structure inherited from the indexing, though the indexing itself is not in N.

Theorem

Let N the model constructed above. If T is

1. *$\aleph_0$-categorical strongly minimal, then $\mathcal{A}$ is amorphous.*
2. *$\aleph_0$-categorical $\aleph_0$-stable, then $\mathcal{A}$ has finite MT-rank.*

Let $M \models ZF + V = L$. Assume M contains an infinite set A that admits finitely many relations and models a first order relational theory T.
We add generic sets of reals indexed by A and the set that contains them to M. Let N be the resulting model. Let $\mathcal{A}$ be the set of sets of generic reals in N. $\mathcal{A}$ has the relational structure inherited from the indexing, though the indexing itself is not in N.

Theorem

Let N the model constructed above. If T is

1. *$\aleph_0$-categorical strongly minimal, then $\mathcal{A}$ is amorphous.*
2. *$\aleph_0$-categorical $\aleph_0$-stable, then $\mathcal{A}$ has finite MT-rank.*
3. *$\aleph_0$-categorical o-minimal, then $\mathcal{A}$ is o-amorphous.*

Let $M \models ZF + V = L$. Assume M contains an infinite set A that admits finitely many relations and models a first order relational theory T.
We add generic sets of reals indexed by A and the set that contains them to M. Let N be the resulting model. Let $\mathcal{A}$ be the set of sets of generic reals in N. $\mathcal{A}$ has the relational structure inherited from the indexing, though the indexing itself is not in N.

Theorem

Let N the model constructed above. If T is

1. *$\aleph_0$-categorical strongly minimal, then $\mathcal{A}$ is amorphous.*
2. *$\aleph_0$-categorical $\aleph_0$-stable, then $\mathcal{A}$ has finite MT-rank.*
3. *$\aleph_0$-categorical o-minimal, then $\mathcal{A}$ is o-amorphous.*
4. *$\aleph_0$-categorical, then $\mathcal{A}$ is weakly Dedekind-finite.*